General Solution of the Rigid Body Impact Problem

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Nomenclature

e = coefficient of restitution
t = time
C = velocity of compression at any time
C_0 = initial velocity of compression
F = frictional force imparted to body 1
I_F = impulse imparted to body 1 by frictional force
I_N = impulse imparted to body 1 by normal force
N = normal force imparted to body 1
S = velocity of sliding at any time
S_0 = initial velocity of sliding
μ = coefficient of friction
k = radius of gyration
u,v = components of velocity of c g of the body
x,y = components of distance of c g from point of impact
M = mass
ω = angular velocity

Subscripts and superscripts

1 = refers to body 1
2 = refers to body 2
( ) = value before impact
( )′ = value during or after impact

Introduction

In solving many technical problems involving impact,[1] it is permissible to neglect the frictional forces because the elastic forces between the bodies are much larger and the duration of the impact is very short It is only necessary to consider the degree of elasticity of the impact Therefore, the solution of these problems is considerably simplified

In the problem of impact of two rigid bodies at relatively small velocities and/or shallow angle of approach, the influence of the friction becomes significant The presented solution of this problem involves analyzing the entire course of an eccentric impact in a diagram whose coordinates are compression and friction impulses

The general problem of impact was extensively analyzed in the 19th century by Poisson, Coriolis, Phillips, Routh, and others The presented method is based upon the similarly titled analysis performed by Routh[2] in 1860

The method is applicable for analyzing the impact associated with problems such as docking maneuvering of spacecraft, investigations of the lunar surface (determining the coefficients of restitution and elasticity of the lunar surface), skid landing on terrestrial and planetary surfaces, head-on and oblique automobile collisions, etc

For simplicity, we consider the impact of two two-dimensional bodies with arbitrary initial conditions (see Fig 1); the analysis can be extended to three-dimensional bodies The bodies are rigid, except at the immediate vicinity of the impact

Dynamic Equations

The equations of conservation of linear and angular momentum of the two bodies are

$$M_1(u_1' - u_1) = -I_F \quad (1)$$

$$M_1(v_1' - v_1) = I_N \quad (2)$$

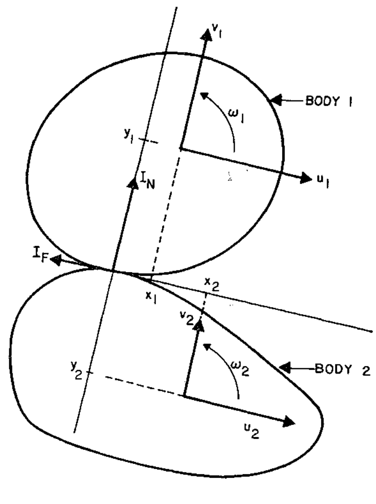


Fig 1 Eccentric impact of bodies

$$M_1k_1^2(\omega_1' - \omega_1) = -I_Fy_1 - I_Nx_1 \quad (3)$$

$$M_2(u_2' - u_2) = I_F \quad (4)$$

$$M_2(v_2' - v_2) = -I_N \quad (5)$$

$$M_2k_2^2(\omega_2' - \omega_2) = I_Nx_2 - I_Fy_2 \quad (6)$$

Equations (1–6) contain eight unknowns: u_1', v_1', ω_1' u_2', v_2', ω_2', I_F, and I_N The total values of I_F and I_N will be determined by a graphical method

The impulse due to normal pressure is given by

$$I_N = \int_{t_0}^{t} Ndt$$

and the coefficient of restitution is

$$e = \int_{t_a}^{t_b} Ndt \Big/ \int_{t_0}^{t_a} Ndt$$

I_N increases throughout the duration of the impact (see Fig 2)

As I_N proceeds from zero to its final value, the impulse due to friction pressure I_F proceeds by either $dI_F = \pm \mu dI_N$, or dI_F is just sufficient to prevent sliding either in the same or in the opposite direction (hence called forward and backward direction) in which the bodies were sliding at the time that impact started The appropriate relationship of I_F to I_N at any time during the impact will be established later in the discussion of the various types of impulse diagrams The coefficient of friction μ will be considered throughout this report to have initially a higher value μ_{static} and gradually change to a constant value of the (dynamic) coefficient μ, although the method does not impose this restriction

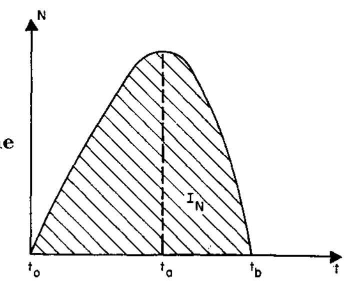
Velocity of Sliding and Compression

The velocities of sliding and compression at any time during, or at the end of the impact, are obtained considering the translation and rotation of the impacting bodies:

$$S = u_1' + \omega_1'y_1 - u_2' + \omega_2'y_2 \quad (7)$$

$$C = v_2' - \omega_2'x_2 - v_1' + \omega_1'x_1 \quad (8)$$

Fig 2 Normal force vs time



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The initial velocities of sliding and compression are determined by the initial conditions, and we have

$$S_0 = u_1 + \omega_1 y_1 - u_2 + \omega_2 y_2 \tag{9}$$

$$C_0 = v_2 - \omega_2 x_2 - v_1 + \omega_1 x_1 \tag{10}$$

Substituting Eqs (1–6, 9, and 10) into (7) and (8), we get

$$S = S_0 - aI_F - cI_N \tag{11}$$

$$C = C_0 - cI_F - bI_N \tag{12}$$

where we introduced the following constants that are independent of the initial motion:

$$\left.\begin{aligned} a &= \frac{1}{M_1} + \frac{1}{M_2} + \frac{y_1^2}{M_1k_1^2} + \frac{y_2^2}{M_2k_2^2} \\ b &= \frac{1}{M_1} + \frac{1}{M_2} + \frac{x_1^2}{M_1k_1^2} + \frac{x_2^2}{M_2k_2^2} \\ c &= \frac{x_1y_1}{M_1k_1^2} - \frac{x_2y_2}{M_2k_2^2} \end{aligned}\right\} \tag{13}$$

Impulse Diagrams

The graphical method for determining the total normal and frictional impulse consists of analyzing the entire course of the impact in a diagram with coordinates I_N and I_F (hence called impulse diagram) The relation between I_N and I_F at any time during the impact is given by the representative point in Fig 3 At the beginning of the impact, the representative point is at the origin From this location it proceeds along a path whose tangent, in relation to the I_N axis, is equal to the coefficient of friction The various possible paths of the representative point, beyond the point indicated in the figure, will be discussed in the next section The representative point can move only upward in the diagram, since I_N increases with time throughout the duration of the impact, even when the force N decreases, as shown in Fig 2 On the other hand, the representative point can move at any time in the positive or negative I_F direction corresponding to the direction of the frictional force

By setting the left sides of Eqs (11) and (12) equal to zero, we obtain the equations of the no-sliding line and of the line of maximum compression:

$$I_N = (S_0/c) - (a/c)I_F \tag{14}$$

$$I_N = (C_0/b) - (c/b)I_F \tag{15}$$

The slopes of the lines defined by Eqs (14) and (15) depend upon the constants of the system given in (13) and are independent of coefficient of friction of the system The no-sliding line is always steeper than the line of maximum compression The no-sliding line divides the diagram into regions of "forward" and "backward" sliding Since the representative point can move only upward, it can cross the no-sliding line only once during the impact, i e , the direction of sliding of the bodies can reverse only once at the most This is in agreement with energy considerations

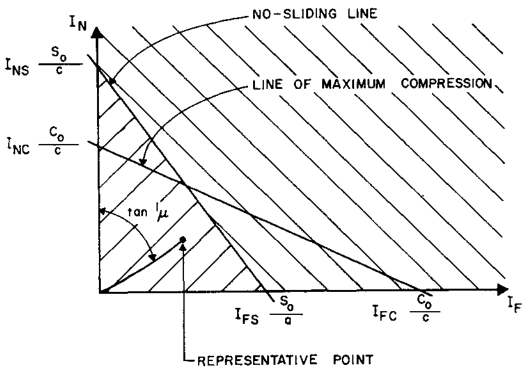

Fig 3 Impulse diagram: ///// region of bodies sliding in "forward" direction; \\\\\ region of bodies sliding in "backward" direction

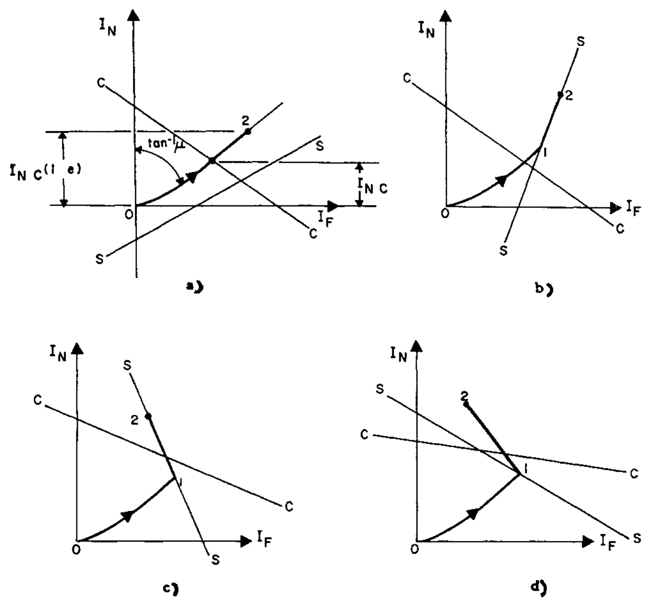

Fig 4 Types of impulse diagrams (S-S no sliding line; C C line of maximum compression; and 0-2 path of the representative point): a) impacting bodies slide in "forward" direction through the duration of the impact; b) 0-1 sliding in forward direction, 1-2 friction prevents tendency to sliding in forward direction; c) 0-1 sliding in forward direction, 1-2 friction prevents tendency to sliding in forward direction; d) 0-1 sliding in forward direction, 1-2 sliding in backward direction

Types of Impulse Diagrams

In the following, we consider the path of a representative point beyond the point that was discussed in conjunction with Fig 3 and its relationship to the lines of no-skidding and of maximum compression

If the representative point does not intersect the "no-sliding line," as shown in Fig 4a, then the direction of sliding is the same throughout the impact This situation is encountered in the analysis of phenomena such as skidding, landing, and oblique automobile collisions The termination of the impact occurs when I_N is equal to $(1 + e)$ multiplied by the value of I_N at the intersection of the path of the representative point and the line of maximum compression The coordinates I_N and I_F of the point corresponding to the termination of the impact are the total impulse communicated to body 1 by normal and frictional forces

Alternatively, if the path of the representative point intersects the "no-sliding line," it proceeds therein from one of the two possible paths

1) Friction necessary to prevent sliding in either forward or reverse direction is less than sliding friction μ, and the bodies continue to roll (Figs 4b and 4c) The difference between these cases is the section 1-2 of the path; the value of I_F continues to increase in Fig 4b, whereas in Fig 4c, I_F decreases In other words, friction prevents the tendency to sliding in the first case in the forward direction, in the latter case in the backward direction

2) The sliding of the bodies reverses since friction necessary to prevent sliding in the backward direction is larger than sliding friction

The representative points corresponding to the termination of the impacts described in Figs 4b, 4c, and 4d are found in the same way as discussed in the case of those described in Fig 4a

Solution of the Dynamic Equations

The linear and the angular velocities of the bodies just after the impact can be calculated from Eqs (1–6), using the values of I_N and I_F determined by the graphical method

The graphical method and the solution of the equations are suitable for programming on a digital computer, and, thereby, it is possible to examine rapidly a large number of possible impacts

References

[1] Timoshenko, S and Young, D H *Advanced Dynamics* (McGraw-Hill Book Co, Inc, New York 1948)

[2] Routh E J *The Elementary Part of Dynamics of a System of Rigid Bodies* (The Macmillan Co, New York, 1905)

A Note on "Pressure Distribution for Hypersonic Boundary-Layer Flow"

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THE pressure distribution of Ref 3 is improved here to represent, more accurately, the two cases of strong and weak hypersonic boundary-layer interaction phenomena for the derived flat-plate case from the wedge Incidentally, this also happens to be the pressure distribution of Hayes and Probstein[2] for the inviscid case The suitability of this pressure distribution for the viscous case can also be seen here

Analysis

From shock relations, we have

$$\frac{P_2 - P_1}{P_1} = \frac{2\gamma}{\gamma + 1}(M_1^2 \sin^2\theta - 1) \tag{1}$$

$$(M_1^2 \sin^2\theta - 1) = \frac{\gamma + 1}{2} M_1^2 \frac{\sin\theta \sin\Delta}{\cos(\theta - \Delta)} \tag{2}$$

$$\sin\Delta = (1 - \epsilon)\sin\theta\cos(\theta - \Delta) \tag{3}$$

where

P_2 = pressure behind the shock wave
P_1 = freestream pressure
γ = specific heats ratio
M_1 = freestream Mach number
θ = shock angle
Δ = deflection angle
ϵ = density ratio = $(\gamma - 1)/(\gamma + 1) + 2/[(\gamma + 1) M_1^2 \times \sin^2\theta]$

To satisfy the asymptotic conditions at $x = +\infty$ (x being the surface coordinate), let

$$\left.\begin{aligned}\theta &= \theta_0 + \theta_1\\ \Delta &= \Delta_0 + \Delta_1\end{aligned}\right\} \tag{4}$$

where θ_0 and Δ_0 are shock and deflection angles for the inviscid case, and θ_1 and Δ_1 correspond to shock and deflection angles for the viscous case; and so, after approximating $\cos(\theta - \Delta) \approx 1$, we have from relations (1–4)

$$\frac{P_2^1}{P_1} - 1 = \frac{\gamma M_1^2 \sin^2(\Delta_0 + \Delta_1)}{1 - \left\{\frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_1^2 \sin^2(\theta_0 + \theta_1)}\right\}} \tag{5}$$

where P_2^1 is the pressure on the boundary layer

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The pressure distribution in relation (5) for the inviscid case, as represented by Hayes and Probstein,[2] is

$$\frac{P_2}{P_1} = 1 + \frac{\gamma M_1^2 \sin^2\Delta}{(1 - \epsilon)} \tag{6}$$

Before discussing the pressure distribution (5) further, we derive here the relations between θ_1 and Δ_1 for the special flat-plate case, placed along the freestream direction, from relation (3)

For the viscous case [Eq (3)], we have

$$(\Delta_0 + \Delta_1) = (\theta_0 + \theta_1) \times \left[1 - \frac{\gamma - 1}{\gamma + 1} - \frac{2}{(\gamma + 1)M_1^2(\theta_0^2 + 2\theta_0\theta_1 + \theta_1^2)}\right] \tag{7}$$

where

$$\sin(\Delta_0 + \Delta_1) \approx (\Delta_0 + \Delta_1)$$

$$\sin(\theta_0 + \theta_1) \approx (\theta_0 + \theta_1)$$

Relation (7), for the special case of $\theta_0 \approx \sin^{-1}(1/M_1)$ and $\Delta_0 \approx 0$, gives

$$M_1\Delta_1 = (1 + M_1\theta_1) \times \left[\frac{2}{\gamma + 1} - \frac{2}{(\gamma + 1)(1 + 2M_1\theta_1 + M_1^2\theta_1^2)}\right]$$

$$= (1 + M_1\theta_1)\left\{\frac{2}{\gamma + 1}\left[1 - \frac{1}{(1 + 2M_1\theta_1 + M_1^2\theta_1^2)}\right]\right\} \tag{8}$$

For the strong interaction case of $M_1\theta_1 \gg 1$, we have from relation (8) that

$$\Delta_1 \approx [2/(\gamma + 1)]\theta_1 \tag{9}$$

For the weak interaction case of $M_1\theta_1 \ll 1$, we have from relation (8)

$$M_1\Delta_1 = (1 + M_1\theta_1)\left\{\frac{2}{\gamma + 1} \times [1 - (1 - 2M_1\theta_1 - M_1^2\theta_1^2 + \quad)]\right\}$$

$$= (1 + M_1\theta_1)\left\{\frac{2}{\gamma + 1} M_1\theta_1[2 + M_1\theta_1 - \quad]\right\}$$

Therefore,

$$\Delta_1 \approx \frac{4}{\gamma + 1}\theta_1 \tag{10}$$

Hence, from pressure distribution (5), we have for the plate case

$$\frac{P_2^1}{P_1} - 1 = \frac{\gamma M_1^2 \Delta_1^2}{[2/(\gamma + 1)]\{1 - [1/(1 + 2M_1\theta_1 + M_1^2\theta_1^2)]\}} \tag{11}$$

For the strong interaction case we have, from relations (9) and (11)

$$\frac{P_2'}{P_1} - 1 = \frac{\gamma k^2}{\frac{2}{\gamma + 1}\left(1 - \frac{1}{\{1 + (\gamma + 1)k + [(\gamma + 1)/2]^2 k^2\}}\right)}$$

where $k = M_1\Delta_1$ and which, for $k \gg 1$, becomes

$$P_2^1/P_1 = 1 + [\gamma(\gamma + 1)/2]k^2 \tag{12}$$

For the weak interactions case, we have, from relations (8)